INTERMEDIATE ARITHMETIC OPERATIONS ON ORDINAL NUMBERS

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Abstract. There are two well-known ways of doing arithmetic with ordinal numbers: the "ordinary" addition, multiplication, and exponentiation, which are defined by transfinite iteration; and the "natural" (or Hessenberg) addition and multiplication (denoted $\oplus$ and $\otimes$), each satisfying its own set of algebraic laws. In 1909, Jacobsthal considered a third, intermediate way of multiplying ordinals (denoted $\times$), defined by transfinite iteration of natural addition, as well as the notion of exponentiation defined by transfinite iteration of his multiplication, which we denote $\alpha^{\times\beta}$. (Jacobsthal's multiplication was later rediscovered by Conway.) Jacobsthal showed these operations too obeyed algebraic laws. In this paper, we pick up where Jacobsthal left off by considering the notion of exponentiation obtained by transfinitely iterating natural multiplication instead; we will denote this $\alpha^{\otimes\beta}$. We show that $\alpha^{\otimes(\beta\oplus\gamma)} = (\alpha^{\otimes\beta}) \otimes (\alpha^{\otimes\gamma})$ and that $\alpha^{\otimes(\beta\times\gamma)} = (\alpha^{\otimes\beta})^{\otimes\gamma}$; note the use of Jacobsthal's multiplication in the latter. We also demonstrate the impossibility of defining a "natural exponentiation" satisfying reasonable algebraic laws.

1. Introduction

In this paper, we introduce a new form of exponentiation of ordinal numbers, which we call *super-Jacobsthal exponentiation*, and study its properties. We show it satisfies two analogues of the usual laws of exponentiation. These laws relate super-Jacobsthal exponentiation to other previously studied operations on the ordinal numbers: natural addition, natural multiplication, and Jacobsthal's multiplication. We also show that there is no "natural exponentiation" analogous to natural addition and natural multiplication.

There are two well-known ways of doing arithmetic with ordinal numbers. Firstly, there are the "ordinary" addition, multiplication, and exponentiation. These are defined by starting with the successor operation S and transfinitely iterating; $\alpha+\beta$ is defined by applying to α the successor operation β-many times; $\alpha\beta$ is α added to itself β-many times; and α^β is α multiplied by itself β-many times. These also have order-theoretic definitions.

The ordinary operations obey some of the usual relations between arithmetic operations:

(1) Associativity of addition: $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
(2) Left-distributivity of multiplication over addition: $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$
(3) Associativity of multiplication: $\alpha(\beta\gamma) = (\alpha\beta)\gamma$
(4) Exponentiation converts addition to multiplication: $\alpha^{\beta+\gamma} = \alpha^\beta\alpha^\gamma$

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(5) Exponential of a product is iterated exponentiation: $\alpha^{\beta\gamma} = (\alpha^\beta)^\gamma$

Note that these operations are not commutative; for instance, $1 + \omega = \omega \neq \omega + 1$ and $2\omega = \omega \neq \omega 2$. Note further that distributivity does not work on the right; for instance,

$$(1 + 1)\omega = \omega \neq \omega 2 = (1\omega) + (1\omega).$$

One can also consider infinitary addition and multiplication, so that

$$\alpha\beta = \sum_{i\in\beta} \alpha$$

and

$$\alpha^\beta = \prod_{i\in\beta} \alpha;$$

these satisfy infinitary analogues of the above laws, though we will not detail this here.

Then there are the "natural" addition and multiplication, sometimes known as the Hessenberg operations [9, pp. 73–81], which we will denote by $\oplus$ and $\otimes$, respectively. These have several equivalent definitions; the simplest definition is in terms of Cantor normal form. Recall that each ordinal number α can be written uniquely as $\omega^{\alpha_0} a_0 + \ldots + \omega^{\alpha_r} a_r$, where $\alpha_0 > \ldots > \alpha_r$ are ordinals and the a_i are positive integers (note that r may be 0); this is known as its Cantor normal form. (We will also sometimes, when it is helpful, write $\alpha = \omega^{\alpha_0} a_0 + \ldots + \omega^{\alpha_r} a_r + a$ where a is a whole number and $\alpha_r > 0$ – that is to say, we will sometimes consider the finite part of α separately from the rest of the Cantor normal form.) Then natural addition and multiplication can roughly be described as adding and multiplying Cantor normal forms as if these were "polynomials in ω". More formally:

Definition 1.1. The *natural sum* of two ordinals α and β, here denoted $\alpha \oplus \beta$, is defined by adding up their Cantor normal forms as if they were "polynomials in ω". That is to say, if there are ordinals $\gamma_0 > \ldots > \gamma_r$ and whole numbers $a_0, \ldots, a_r$ and $b_0, \ldots, b_r$ such that $\alpha = \omega^{\gamma_0} a_0 + \ldots + \omega^{\gamma_r} a_r$ and $\beta = \omega^{\gamma_0} b_0 + \ldots + \omega^{\gamma_r} b_r$, then

$$\alpha \oplus \beta = \omega^{\gamma_0}(a_0 + b_0) + \ldots + \omega^{\gamma_r}(a_r + b_r).$$

Definition 1.2. The *natural product* of α and β, here denoted $\alpha \otimes \beta$, is defined by multiplying their Cantor normal forms as if they were "polynomials in ω", using the natural sum to add the exponents. That is to say, if we write $\alpha = \omega^{\alpha_0} a_0 + \ldots + \omega^{\alpha_r} a_r$ and $\beta = \omega^{\beta_0} b_0 + \ldots + \omega^{\beta_s} b_s$ with $\alpha_0 > \ldots > \alpha_r$ and $\beta_0 > \ldots > \beta_s$ ordinals and the a_i and b_i positive integers, then

$$\alpha \otimes \beta = \bigoplus_{\substack{0\le i\le r\\ 0\le j\le s}} \omega^{\alpha_i \oplus \beta_j} a_i b_j.$$

The natural operations also have recursive definitions, due to Conway [5, pp. 3–14]. Let us use the following notation:

Notation 1.3. If S is a set of ordinals, $\sup' S$ will denote the smallest ordinal greater than all elements of S. (This is equal to $\sup(S+1)$; it is also equal to $\sup S$ unless S has a greatest element, in which case it is $(\sup S) + 1$.)

Then these operations may be characterized by:

Theorem 1.4 (Conway)**.** *We have:*

(1) For ordinals α and β,

$$\alpha \oplus \beta = \sup'(\{\alpha \oplus \beta' : \beta' < \beta\} \cup \{\alpha' \oplus \beta : \alpha' < \alpha\})$$

(2) For ordinals α and β,

$$\alpha \otimes \beta = \min\{x : x \oplus (\alpha' \otimes \beta') > (\alpha \otimes \beta') \oplus (\alpha' \otimes \beta) \text{ for all } \alpha' < \alpha \text{ and } \beta' < \beta\}$$

These operations have some nicer algebraic properties than the ordinary operations – they are commutative, and have appropriate cancellation properties; indeed, a copy of the ordinals with the natural operations embeds in the field of surreal numbers. However, not being defined by transfinite iteration, these operations are not continuous in either operand, whereas the ordinary operations are continuous in the right operand. These operations also have order-theoretic definitions, due to Carruth [4]; see De Jongh and Parikh [6] for more on this.

The natural operations also obey algebraic laws:

(1) Associativity of addition: $\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$
(2) Distributivity of multiplication over addition: $\alpha \otimes (\beta \oplus \gamma) = (\alpha \otimes \beta) \oplus (\alpha \otimes \gamma)$
(3) Associativity of multiplication: $\alpha \otimes (\beta \otimes \gamma) = (\alpha \otimes \beta) \otimes \gamma$

Moreover, both $\oplus$ and $\otimes$ are commutative, as mentioned above. As such, $\otimes$ in fact distributes over $\oplus$ on both sides, not just on the left.

There is no known definition of a "natural exponentiation", and in Theorem 1.12 we will show no such operation can exist.

In 1909, E. Jacobsthal introduced [11] another sort of multiplication and exponentiation for ordinals. He defined the operation $\times$, which we will refer to as "Jacobsthal multiplication", by transfinitely iterating natural addition; $\alpha \times \beta$ means α added to itself β-many times, using natural addition. More formally:

Definition 1.5 (Jacobsthal)**.** We define the operation $\times$ by

(1) For any α, $\alpha \times 0 := 0$.
(2) For any α and β, $\alpha \times (S\beta) := (\alpha \times \beta) \oplus \alpha$.
(3) If β is a limit ordinal, $\alpha \times \beta := \lim_{\gamma < \beta} (\alpha \times \gamma)$.

This multiplication is not commutative; for instance, $2 \times \omega = \omega \neq \omega 2 = \omega \times 2$. We will discuss other algebraic laws for it shortly.

Jacobsthal's multiplication was later rediscovered by Conway and discussed by Gonshor [8] and by Hickman [10]; as such it has also been referred to as "Conway multiplication", though this name is used also of other operations. It was also later rediscovered by Abraham and Bonnet [1].

Jacobsthal multiplication is intermediate between ordinary and natural multiplication; in particular, one has the inequalities

$$\alpha\beta \le \alpha \times \beta \le \alpha \otimes \beta$$

for any ordinals α and β.

Jacobsthal then went on to describe a notion of exponentiation obtained by transfinitely iterating $\times$; we will refer to it as "Jacobsthal exponentiation". He denoted it by $\alpha^{\underline{\beta}}$, but we will denote it by $\alpha^{\times\beta}$. More formally:

Definition 1.6 (Jacobsthal)**.** We define $\alpha^{\times\beta}$ by

(1) For any α, $\alpha^{\times 0} := 1$.
(2) For any α and β, $\alpha^{\times(S\beta)} := (\alpha^{\times\beta}) \times \alpha$.
(3) If β is a limit ordinal, $\alpha^{\times\beta} := \lim_{\gamma < \beta} (\alpha^{\times\gamma})$.

Jacobsthal then proved [11] the algebraic law:

Theorem 1.7 (Jacobsthal). *For any ordinals α, β, and γ, one has*

$$\alpha \times (\beta \oplus \gamma) = (\alpha \times \beta) \oplus (\alpha \times \gamma).$$

That is to say, $\times$ left-distributes over $\oplus$.

Jacobsthal gave only a computational proof of this law; in this paper we will provide a more straightforward proof by transfinite induction. Note also that this distributivity works only on the left and not on the right; for instance,

$$(1 \oplus 1) \times \omega = \omega \neq \omega 2 = (1 \times \omega) \oplus (1 \times \omega).$$

Regardless, once one has this in hand, it is straightforward to prove by transfinite induction that

Theorem 1.8 (Jacobsthal). *The following algebraic relations hold:*

(1) *Jacobsthal multiplication is associative: For any α, β, and γ, one has*
$$\alpha \times (\beta \times \gamma) = (\alpha \times \beta) \times \gamma.$$
(2) *Jacobsthal exponentiation converts ordinary addition to Jacobsthal multiplication: For any α, β, and γ, one has*
$$\alpha^{\times(\beta+\gamma)} = (\alpha^{\times\beta}) \times (\alpha^{\times\gamma}).$$
(3) *The Jacobsthal exponential of an ordinary product is an iterated Jacobsthal exponentiation: For any α, β, and γ, one has*
$$\alpha^{\times(\beta\gamma)} = (\alpha^{\times\beta})^{\times\gamma}.$$

One can consider more generally infinitary versions of these addition and multiplication operations, where for instance $\bigoplus_{i<\beta} \alpha_i$ is defined in the obvious way, so that $\alpha \times \beta = \bigoplus_{i<\beta} \alpha$. Some care is warranted, however; as the natural operations are not continuous in the right operand, $1 \oplus (1 \oplus 1 \oplus \ldots)$ is not equal to $1 \oplus 1 \oplus \ldots$ (as $\omega + 1 \neq \omega$), and neither is $2 \otimes (2 \otimes 2 \otimes \ldots)$ equal to $2 \otimes 2 \otimes \ldots$ (as $\omega 2 \neq \omega$). Moreover, while the finitary natural operations have order-theoretic meaning [6], and one was recently discovered for the infinitary natural sum [13, 14], none is known for the infinitary natural product.

Regardless, with these, one may consider infinitary versions of these relations, though these are not our main focus and we will not list them all here. Notably, however, the same reasoning that leads to the relation $\alpha \times (\beta \times \gamma) = (\alpha \times \beta) \times \gamma$ shows more generally that

$$\alpha \times \bigoplus_{i<\gamma} \beta_i = \bigoplus_{i<\gamma} (\alpha \times \beta_i);$$

the associativity of Jacobsthal multiplication is just the case where all the β_i are equal.

1.1. **Main results.** Jacobsthal's multiplication and exponentation both work quite nicely algebraically – the multiplication is associative, and the exponentation obeys versions of the expected relations.

But there is yet another way we could define a notion of exponentiation on ordinals; instead of transfinitely iterating ordinary multiplication or Jacobsthal multiplication, we could transfinitely iterate natural multiplication. This leads us to introduce the following operation:

Definition 1.9. We define $\alpha^{\otimes\beta}$ by

(1) For any α, $\alpha^{\otimes 0} := 1$.
(2) For any α and β, $\alpha^{\otimes(S\beta)} := (\alpha^{\otimes\beta}) \otimes \alpha$.
(3) If β is a limit ordinal, $\alpha^{\otimes\beta} := \lim_{\gamma<\beta}(\alpha^{\otimes\gamma})$.

We refer to this operation as "super-Jacobsthal exponentiation". It was considered briefly by De Jongh and Parikh [6] but has otherwise been mostly unexplored.

With this in hand, we can now state the following relation, which we will prove in Section 3:

Theorem 1.10. *For any ordinals α, β, and γ, one has*

$$\alpha^{\otimes(\beta\oplus\gamma)} = (\alpha^{\otimes\beta}) \otimes (\alpha^{\otimes\gamma}).$$

That is to say, super-Jacobsthal exponentiation converts natural addition to natural multiplication.

With this theorem, it is straightforward to prove by transfinite induction that

Theorem 1.11. *For any ordinals α, β, and γ, one has*

$$\alpha^{\otimes(\beta\times\gamma)} = (\alpha^{\otimes\beta})^{\otimes\gamma}.$$

That is to say, the super-Jacobsthal exponential of a Jacobsthal product is an iterated super-Jacobsthal exponential.

More generally, given ordinals α and γ and a family of ordinals β_i indexed by γ, one has

$$\alpha^{\otimes(\bigoplus_{i<\gamma}\beta_i)} = \bigotimes_{i<\gamma} \alpha^{\otimes\beta_i}.$$

Note the appearance of Jacobsthal multiplication – not ordinary or natural multiplication – on the left hand side of the equation. This occurs because Theorem 1.11 comes from transfinitely iterating Theorem 1.10, and when one transfinitely iterates natural addition, one gets Jacobsthal multiplication.

We also close off the possibility of there existing a "natural exponentiation". Let us denote such a thing by $e(\alpha,\beta)$, where α is the base and β is the exponent. Then:

Theorem 1.12. *There is no natural exponentiation $e(\alpha,\beta)$ on the ordinals satisfying the following conditions:*

(1) $e(\alpha,1) = \alpha$
(2) For $\alpha > 0$, $e(\alpha,\beta)$ is weakly increasing in β.
(3) $e(\alpha,\beta)$ is weakly increasing in α.
(4) $e(\alpha,\beta\oplus\gamma) = e(\alpha,\beta)\otimes e(\alpha,\gamma)$
(5) $e(\alpha,\beta\otimes\gamma) = e(e(\alpha,\beta),\gamma)$

The same holds if hypothesis (5) is replaced with the following hypothesis (5'): $e(\alpha\otimes\beta,\gamma) = e(\alpha,\gamma)\otimes e(\alpha,\gamma)$.

Remark 1.13. The version of this theorem where hypothesis (5') is used was also proven independently, in slightly stronger form, by Asperó and Tsaprounis [2], using essentially the same means.

Note here that since addition and multiplication in the surreals agree with natural addition and natural multiplication on the ordinals, one might attempt to define a "natural exponentiation" based on the theory of of surreal exponentiation (developed by Gonshor [7, pp. 143–190]) via the map $(\alpha,\beta) \mapsto \exp(\beta\log\alpha)$. However,

while this makes perfect sense for surreal numbers, the ordinals are not closed under this operation; it turns out that, using the usual notation for surreal numbers, one has

$$\exp(\omega \log \omega) = \omega^{\omega^{1+1/\omega}},$$

which is not an ordinal. One could attempt to remedy this by rounding up to the next ordinal, but the resulting operation is lacking in algebraic laws.

Because there are a number of similar but slightly different operations being considered, the reader may want to consult Table 1 on page 6 as a helpful guide to the operations, and Table 2 on page 7 as a guide to their algebraic laws.

TABLE 1. Each operation is the transfinite iteration of the one above it, yielding three vertical families of operations, in addition to the diagonal family of natural operations; surreal exponentiation is dashed-out because it is not actually an operation on ordinals, and natural exponentiation does not exist. Each operation not on the diagonal, being a transfinite iteration, is continuous in β. In addition, each operation is pointwise less-than-or-equal-to those on its right; see Appendix A.

Natural operations $\searrow$	S-based $\downarrow$	$\oplus$-based $\downarrow$	$\otimes$-based $\downarrow$	
Successor	Successor $S\alpha$			
Addition	Ordinary $\alpha + \beta$	Natural $\alpha \oplus \beta$		
Multiplication	Ordinary $\alpha\beta$	Jacobsthal $\alpha \times \beta$	Natural $\alpha \otimes \beta$	
Exponentiation	Ordinary α^{β}	Jacobsthal $\alpha^{\times\beta}$	Super-J. $\alpha^{\otimes\beta}$	Surreal $\exp(\beta \log \alpha)$

1.2. **Discussion.** Before we go on to prove the main theorem, let us make a note about one reason that Theorems 1.7 and 1.10 might be considered unexpected.

Let us consider ways in which we can interpret, and thus begin to prove, these statements. One possible interpretation for statements about ordinal operations is to read them as statements about operations on order types. The other possibility, which seems more appropriate here, is, to read them as statements about transfinitely iterating things – which is, after all, what they are. This interpretation suggests approaching the problem via transfinite induction.

However, this approach too is misleading. Let us consider Theorem 1.7, which states that $\alpha \times (\beta \oplus \gamma) = (\alpha \times \beta) \oplus (\alpha \times \gamma)$. Taking this approach would indeed suggest a relation of this form – but not this relation. Rather, it suggests the false statement

$$\alpha \times (\beta + \gamma) = (\alpha \times \beta) \oplus (\alpha \times \gamma).$$

Table 2. A table of the algebraic laws described above. Each law has been placed into one of the three vertical families in Table 1 based on the "main" operation involved, i.e., whichever one is in the bottom-most row in Table 1 – note that many of these laws relate operations in different vertical families, and so would go in more than one column without this choice of convention. In addition, the operations $\oplus$ and $\otimes$ are both commutative, but this is not listed here as it does not fit into any of the patterns displayed here.

Successor-based	**$\oplus$-based**	**$\otimes$-based**
$\alpha+(\beta+\gamma)=(\alpha+\beta)+\gamma$	$\alpha\oplus(\beta\oplus\gamma)=(\alpha\oplus\beta)\oplus\gamma$	Not applicable
$\alpha(\beta+\gamma)=\alpha\beta+\alpha\gamma$	$\alpha\times(\beta\oplus\gamma)=(\alpha\times\beta)\oplus(\alpha\times\gamma)$	$\alpha\otimes(\beta\oplus\gamma)=(\alpha\otimes\beta)\oplus(\alpha\otimes\gamma)$
$\alpha(\beta\gamma)=(\alpha\beta)\gamma$	$\alpha\times(\beta\times\gamma)=(\alpha\times\beta)\times\gamma$	$\alpha\otimes(\beta\otimes\gamma)=(\alpha\otimes\beta)\otimes\gamma$
$\alpha^{\beta+\gamma}=\alpha^\beta\alpha^\gamma$	$\alpha^{\times(\beta+\gamma)}=(\alpha^{\times\beta})\times(\alpha^{\times\gamma})$	$\alpha^{\otimes(\beta\oplus\gamma)}=(\alpha^{\otimes\beta})\otimes(\alpha^{\otimes\gamma})$
$\alpha^{\beta\gamma}=(\alpha^\beta)^\gamma$	$\alpha^{\times(\beta\gamma)}=(\alpha^{\times\beta})^{\times\gamma}$	$\alpha^{\otimes(\beta\times\gamma)}=(\alpha^{\otimes\beta})^{\otimes\gamma}$

After all, $\alpha\times(\beta+\gamma)$ means α added to itself $(\beta+\gamma)$-many times, via the operation $\oplus$; so it would seem this should be the same as the natural sum of α added to itself β-many times, and α added to itself γ-many times.

But this statement is false; a simple counterexample is that $1\times(1+\omega)=\omega$, but $(1\times 1)\oplus(1\times\omega)=\omega+1$. What went wrong? If one tries to turn the above intuition into a proof, analogous to the proof via transfinite induction that $\alpha(\beta+\gamma)=\alpha\beta+\alpha\gamma$, one finds that the proof would require $\alpha\oplus\beta$ to be continuous in β, which is, of course, not the case. Alternatively, one could note that the distributivity of ordinary multiplication over ordinary addition can be viewed as a special case of infinitary ordinary addition satisfying "generalized associativity"; but as noted earlier, since natural addition is not continuous in the right operand, it does not.

And yet, despite this, a law of this form does hold – it's just that the left hand side, instead of $\alpha\times(\beta+\gamma)$, is $\alpha\times(\beta\oplus\gamma)$. This is, first off, a strange-looking expression. It seems (not necessarily correctly) relatively clear what it means to do something $(\beta+\gamma)$-many times; but it's not at all clear what it means to do something $(\beta\oplus\gamma)$-many times. And while the distributivity of ordinary multiplication over ordinary addition comes from "generalized associativity" for infinitary ordinary addition, there is no such infinitary law for this form of distributivity to come from.

In addition, while Theorem 1.7 is undeniably true, Jacobsthal's proof is less than enlightening. Jacobsthal proved this statement by, in essence, computation – showing that the two sides of the equation have the same Cantor normal form. Specifically, he proved it by first proving the following theorem, on how $\alpha\times\beta$ may be computed in Cantor normal form:

Theorem 1.14 (Jacobsthal)**.** *Let α and β be ordinals. Write α in Cantor normal form as*

$$\alpha=\omega^{\alpha_0}a_0+\ldots+\omega^{\alpha_r}a_r;$$

here $\alpha_0,\ldots,\alpha_r$ is a decreasing (possibly empty) sequence of ordinals and the a_i are positive integers. Write β in Cantor normal form as

$$\beta=\omega^{\beta_0}b_0+\ldots+\omega^{\beta_s}b_s+b;$$

here $\beta_0, \ldots, \beta_s$ is a decreasing (possibly empty) sequences of nonzero ordinals, the b_i are positive integers, and b is a nonnegative integer. Then

$$\alpha \times \beta = \omega^{\alpha_0+\beta_0} b_0 + \ldots + \omega^{\alpha_0+\beta_s} b_s + \omega^{\alpha_0}(a_0 b) + \ldots + \omega^{\alpha_r}(a_r b).$$

In other words, if $\beta = \beta' + b$ where β' is either 0 or a limit ordinal and b is finite, then

$$\alpha \times \beta = \omega^{\alpha_0}\beta' + \alpha \times b.$$

With this in hand, Theorem 1.7 is straightforward, but as an explanation, it is not very satisfying.

There is a certain mystery to part (1) of Theorem 1.8 as well. While the proof is a simple transfinite induction using Theorem 1.7, the statement itself still looks strange; why should the operation of $\times$ be associative? Typically, when we prove that an operation $*$ is associative, we are not just proving that $a*(b*c) = (a*b)*c$; rather, we usually do it by proving that $a*(b*c)$ and $(a*b)*c$ are both equal to some object $a*b*c$, and that indeed $a_1 * \ldots * a_r$ makes sense for any finite r – not just proving that this makes sense *because* $*$ happens to be associative, but that this makes sense as a thing on its own, and that this is *why* $*$ must be associative. By contrast, Jacobsthal's proof of the associativity of $\times$ is just that – a proof that $\alpha \times (\beta \times \gamma) = (\alpha \times \beta) \times \gamma$, and no more. It remains unclear how the triple product $\alpha \times \beta \times \gamma$ might be interpreted on its own, without first choosing a parenthesization.

Theorem 1.10 is strange, then, for the same reasons that Theorem 1.7 is. No order-theoretic interpretation is evident, and the "do it this many times" heuristic would again suggest a false statement, this time

$$\alpha^{\otimes(\beta+\gamma)} = (\alpha^{\otimes\beta}) \otimes (\alpha^{\otimes\gamma}),$$

which has the counterexample that $2^{\otimes(1+\omega)} = \omega$ while $(2^{\otimes 1}) \otimes (2^{\otimes\omega}) = \omega 2$, and whose proof fails because $\alpha \otimes \beta$ is not continuous in β. Once again there is the question of what it means to do something $(\beta \oplus \gamma)$-many times. And while the relation $\alpha^{\beta+\gamma}$ comes from "generalized associativity" for (infinitary) ordinary multiplication, and the relation $\alpha^{\times(\beta+\gamma)} = \alpha^{\times\beta} \times \alpha^{\times\gamma}$ comes from the same for (infinitary) Jacobsthal multiplication, there is no such infinitary relation about natural multiplication that would give rise to Theorem 1.10, because $\alpha \otimes \beta$, unlike $\alpha \times \beta$ or $\alpha\beta$, is not continuous in β.

Now, both Theorem 1.7 and Theorem 1.10 can be proved by transfinite induction, as we will do here. Still, these theorems raise some questions:

Question 1.15. *(1) Can Theorem 1.7 be proven by giving an order-theoretic interpretation to both sides?*

(2) Can the associativity of Jacobsthal multiplication be proven by finding a natural way of interpreting $\alpha \times \beta \times \gamma$ without first inserting parentheses?

(3) Can Theorem 1.10 be proven by giving an order-theoretic interpretation to both sides?

P. Lipparini recently found an order-theoretic interpretation for the infinitary natural sum [14], and therefore, implicitly, for Jacobsthal's multiplication, so an answer to questions (1) and (2) may be close at hand. However, his order-theoretic interpretation does not make the problem trivial, so the question remains open. Meanwhile, answering question (3) would require first finding an order-theoretic interpretation for super-Jacobsthal exponentiation in the first place. In addition, this paper's proof of Theorem 1.10 does have some "computational" parts; specifically,

the case when the base α is finite, which relies on the computational Lemma 3.5. A more unified proof that does not need to separate out this case and handle it computationally would also be a good thing.

Before we go on, let us discuss the question of "natural exponentiation". We have here several families of operations, defined by transfinite iteration (the vertical families in Table 1). We can start from the successor operation, and get ordinary addition, ordinary multiplication, and ordinary exponentiation. We can start from natural addition, and get Jacobsthal multiplication and Jacobsthal exponentiation. Or we can start from natural multiplication and get the super-Jacobsthal exponentiation considered here. (One could continue any of these sequences further, into higher hyper operations, as discussed in [3, pp. 66–79], but we will not discuss that possibility here for several reasons, among them that higher hyper operations lack algebraic properties.) Can we continue further the sequence of natural operations (the diagonal family in Table 1), and get a natural exponentiation?

Theorem 1.12 shows that such an operation is not possible, unless one is willing to abandon what should be basic properties of such an operation. One could produce a whole list of conditions that such an operation might be expected to satisfy, for instance:

(1) $e(\alpha, 0) = 1$
(2) $e(\alpha, 1) = \alpha$
(3) $e(0, \alpha) = 0$ for $\alpha > 0$
(4) $e(1, \alpha) = 1$
(5) For $\alpha > 1$, $e(\alpha, \beta)$ is strictly increasing in β.
(6) For $\beta > 0$, $e(\alpha, \beta)$ is strictly increasing in α.
(7) $e(\alpha, \beta \oplus \gamma) = e(\alpha, \beta) \otimes e(\alpha, \gamma)$
(8) $e(\alpha, \beta \otimes \gamma) = e(e(\alpha, \beta), \gamma)$
(9) $e(\alpha \otimes \beta, \gamma) = e(\alpha, \gamma) \otimes e(\beta, \gamma)$
(10) $e(2, \alpha) > \alpha$

But even only a small number of these is enough to cause a contradiction. On the other hand, in the surreals, the exponentiation operation defined by $e(\alpha, \beta) = \exp(\beta \log \alpha)$ for $\alpha > 0$, $e(0, \beta) = 0$ for $\beta > 0$ and $e(0, 0) = 1$ (and left undefined for other inputs) satisfies each of these laws when all terms involved are defined.

To go in a different direction, rather than restricting surreal operations to the ordinals, or trying to define a natural exponentiation on the ordinals analogous to surreal exponentiation, one could also attempt to extend the ordinary ordinal operations, or these intermediate ones, to the surreal numbers. This was accomplished for ordinary addition by Conway [5, Ch. 15]; indeed, he extended it to all games, not just numbers. For ordinary multiplication, there is a definition of S. Norton which was proven by P. Keddie [12] to work for surreal numbers written in a particular form, namely, written with "no reversible options"; see his paper for more. It remains to be seen whether this can be done for Jacobsthal multiplication, or for any of the exponentiation operations considered here; Keddie [12] gives reasons why this may be difficult for exponentiation.

Finally, it is worth noting that all the notions of multiplication and exponentiation considered here are in fact different. An example is provided by considering

$(\omega+2)(\omega+2)$, or $(\omega+2)^2$, since one has the equations

$$\begin{aligned}(\omega+2)^2 &= \omega^2+\omega 2+2,\\ (\omega+2)^{\times 2} &= \omega^2+\omega 2+4,\\ (\omega+2)^{\otimes 2} &= \omega^2+\omega 4+4.\end{aligned}$$

2. Proof there is no natural exponentiation

In this section we prove Theorem 1.12.

Proof of Theorem 1.12. Suppose we had such an operation $e(\alpha,\beta)$. Note that hypotheses (1) and (4) together mean that if k is finite and positive, then $e(\alpha,k)=\alpha^{\otimes k}$, and in particular, if n is also finite, then $e(n,k)=n^k$. By hypothesis (2), this means that for $n\ge 2$ we have $e(n,\omega)\ge\omega$. Let us define $\delta=\deg\deg e(2,\omega)$; since $e(2,\omega)$ is infinite, this is well-defined.

Observe also that by hypothesis (5), we have for n and k as above,

$$e(n^k,\alpha)=e(e(n,k),\alpha)=e(n,k\otimes\alpha)=e(n,\alpha\otimes k)=e(e(n,\alpha),k)=e(n,\alpha)^{\otimes k}.$$

(If we had used instead the alternate hypothesis (5'), this too would prove that $e(n^k,\alpha)=e(n,\alpha)^{\otimes k}$.)

Given any finite $n\ge 2$, choose some k such that $n\le 2^k$; then by the above and hypothesis (3),

$$e(2,\omega)\le e(n,\omega)\le e(2,\omega)^{\otimes k}$$

and so

$$\deg e(2,\omega)\le\deg e(n,\omega)\le(\deg e(2,\omega))\otimes k$$

and so

$$\deg\deg e(2,\omega)\le\deg\deg e(n,\omega)\le\deg\deg e(2,\omega),$$

i.e., $\deg\deg e(n,\omega)=\delta$.

Thus we may define a function $f:\mathbb{N}\to\mathbb{N}$ by defining $f(n)$ to be the coefficient of ω^δ in the Cantor normal form of $\deg e(n,\omega)$. Then since $e(n^k,\omega)=e(n,\omega)^{\otimes k}$, we have $f(n^k)=kf(n)$. And by the above and hypothesis (3) we have that f is weakly increasing, since $\deg e(n,\omega)$ is weakly increasing and no term of size $\omega^{\delta+1}$ or higher ever appears in any $\deg e(n,\omega)$. Finally, we have that $f(2)\ge 1$.

But no such function can exist; given natural numbers n and m, it follows from the above that

$$\lfloor\log_m n\rfloor f(m)\le f(n)\le\lceil\log_m n\rceil f(m)$$

or in other words that

$$\left\lfloor\frac{\log n}{\log m}\right\rfloor\le\frac{f(n)}{f(m)}\le\left\lceil\frac{\log n}{\log m}\right\rceil.$$

If one takes the above and substitutes in n^k for n, one obtains

$$\left\lfloor k\frac{\log n}{\log m}\right\rfloor\le k\frac{f(n)}{f(m)}\le\left\lceil k\frac{\log n}{\log m}\right\rceil.$$

But in particular, this means that

$$k\frac{\log n}{\log m}-1\le k\frac{f(n)}{f(m)}\le k\frac{\log n}{\log m}+1,$$

or in other words, that

$$\frac{\log n}{\log m} - \frac{1}{k} \le \frac{f(n)}{f(m)} \le \frac{\log n}{\log m} + \frac{1}{k};$$

since this holds for any choice of k, we conclude that

$$\frac{f(n)}{f(m)} = \frac{\log n}{\log m}.$$

But the right hand side may be chosen to be irrational, for instance if $m = 2$ and $n = 3$; thus, the function f cannot exist, and thus neither can our natural exponentiation e. □

Remark 2.1. Note that the only use of hypotheses (1) and (4) was to show that for k a positive integer, $e(\alpha, k) = \alpha^{\otimes k}$, so strictly speaking the the theorem could be stated with (1) and (4) replaced by this single hypothesis.

3. Proofs of algebraic laws

In this section we give an inductive proof of Theorem 1.7 and prove Theorems 1.10 and 1.11.

Inductive proof of Theorem 1.7. We induct on β and γ. If $\beta = 0$ or $\gamma = 0$, the statement is obvious. If γ is a successor, say $\gamma = S\gamma'$, then we have

$$\alpha \times (\beta \oplus \gamma) = \alpha \times (\beta \oplus S\gamma') = \alpha \times S(\beta \oplus \gamma') = (\alpha \times (\beta \oplus \gamma')) \oplus \alpha = \\ (\alpha \times \beta) \oplus (\alpha \times \gamma') \oplus \alpha = (\alpha \times \beta) \oplus (\alpha \times \gamma),$$

as needed. If β is a successor, the proof is similar.

This leaves the case where β and γ are both limit ordinals. Note that in this case, $\beta \oplus \gamma$ is a limit ordinal as well, and that

$$\beta \oplus \gamma = \sup(\{\beta \oplus \gamma' : \gamma' < \gamma\} \cup \{\beta' \oplus \gamma : \beta' < \beta\}).$$

So

$$\begin{aligned} \alpha \times (\beta \oplus \gamma) = \sup\{\alpha \times \delta : \delta < \beta \oplus \gamma\} = \\ \sup(\{\alpha \times (\beta' \oplus \gamma) : \beta' < \beta\} \cup \{\alpha \times (\beta \oplus \gamma') : \gamma' < \gamma\}) = \\ \sup(\{(\alpha \times \beta') \oplus (\alpha \times \gamma) : \beta' < \beta\} \cup \{(\alpha \times \beta) \oplus (\alpha \times \gamma') : \gamma' < \gamma\}). \end{aligned} \tag{3.1}$$

Since $\alpha \times \beta$, $\alpha \times \gamma$, and their natural sum are all limit ordinals as well, we have

$$(\alpha \times \beta) \oplus (\alpha \times \gamma) = \sup(\{\delta \oplus (\alpha \times \gamma) : \delta < \alpha \times \beta\} \cup \{(\alpha \times \beta) \oplus \varepsilon : \varepsilon < \alpha \times \beta\}). \tag{3.2}$$

So we want to show that these two sets we are taking the suprema of (in the final expressions in Equations (3.1) and (3.2)) are cofinal, and thus have equal suprema. The first of these is actually a subset of the second, so it suffices to check that it is cofinal in it. So if $\delta < \alpha \times \beta$, then $\delta \le \alpha \times \beta'$ for some $\beta' < \beta$, so $\delta \oplus (\alpha \times \gamma) \le (\alpha \times \beta') \oplus (\alpha \times \gamma)$; similarly with $\varepsilon < \alpha \times \gamma$.

So our two suprema are equal and $\alpha \times (\beta \oplus \gamma) = (\alpha \times \beta) \oplus (\alpha \times \gamma)$; this proves the theorem. □

Now we prove Theorem 1.10. This will require a tiny bit more setup. First, some notation and two lemmas:

Notation 3.3. For an ordinal $\alpha > 0$, $\deg \alpha$ will denote the largest exponent appearing in the Cantor normal form of α.

Notation 3.4. For an ordinal α which is either 0 or a limit ordinal, $\omega^{-1}\alpha$ will denote the unique ordinal β such that $\alpha = \omega\beta$.

Lemma 3.5. *Suppose $a > 1$ is finite and let β be an ordinal. Write $\beta = \beta' + b$, where β' is 0 or a limit ordinal and b is finite. Then*

$$a^{\otimes\beta} = \omega^{\omega^{-1}\beta'} a^b.$$

Proof. We induct on β. If $\beta = 0$, then both sides are equal to 1. If β is a successor ordinal, say $\beta = S\gamma$, then by the inductive hypothesis,

$$a^{\otimes\gamma} = \omega^{\omega^{-1}\gamma'} a^c,$$

where we write $\gamma = \gamma' + c$ analogously to $\beta = \beta' + b$. As $\beta = S\gamma$, we have $\beta' = \gamma'$ and $b = c + 1$. Thus

$$a^{\otimes\beta} = a^{\otimes\gamma} \otimes a = (\omega^{\omega^{-1}\gamma'} a^c) \otimes a = \omega^{\omega^{-1}\beta'} a^b.$$

If β is a limit ordinal, we have two further cases, depending on whether or not β is of the form $\omega^2\gamma$ for some ordinal γ. If not, then β is of the form $\gamma' + \omega$, where γ' is either 0 or a limit ordinal. This means that β is the limit of $\gamma' + c$ for finite c. So then by the inductive hypothesis,

$$a^{\otimes\beta} = \lim_{c<\omega} (\omega^{\omega^{-1}\gamma'} a^c) = \omega^{\omega^{-1}\gamma'+1} = \omega^{\omega^{-1}\beta'},$$

as required.

If so, then we once again consider $\deg a^{\otimes\beta}$. Since β is of the form $\omega^2\gamma$, β is the limit of all ordinals less than it of the form $\omega\gamma$, i.e., it is the limit of all limit ordinals less than it. And for $\gamma < \beta$ a limit ordinal, by the inductive hypothesis, $\deg a^\gamma = \omega^{-1}\gamma$. So again applying the fact that the deg function is increasing, we have that $\deg a^{\otimes\beta} \geq \omega^{-1}\beta$, i.e., that $a^{\otimes\beta} \geq \omega^{\omega^{-1}\beta}$. (Here we also use the continuity of "division by ω", which follows from the continuity of left-multiplication by ω.) Conversely, for $\gamma < \beta$ with γ a limit ordinal, one has $\omega^{-1}\gamma < \omega^{-1}\beta$, and so $a^{\otimes\gamma} < \omega^{\omega^{-1}\beta}$; thus one has $a^{\otimes\beta} \leq \omega^{\omega^{-1}\beta}$. So we conclude, as needed, that $a^{\otimes\beta} = \omega^{\omega^{-1}\beta}$. This proves the lemma. $\square$

Notation 3.6. For ordinals α and β, $\alpha \ominus \beta$ will denote the smallest γ such that $\beta \oplus \gamma \geq \alpha$. For convenience, we will also define

$$f_{\alpha,\beta}(\alpha', \beta') = ((\alpha \otimes \beta') \oplus (\alpha' \otimes \beta)) \ominus (\alpha' \otimes \beta').$$

Note that with this definition, we can rewrite Conway's definition of $\alpha \otimes \beta$ as

$$\alpha \otimes \beta = \sup{}'\{f_{\alpha,\beta}(\alpha', \beta') : \alpha' < \alpha, \beta' < \beta\}.$$

Lemma 3.7. *For fixed α and β, $f_{\alpha,\beta}(\alpha', \beta')$ is increasing in α' and β'.*

Proof. Observe that $f_{\alpha,\beta}(\alpha', \beta')$ is the smallest ordinal greater than the surreal number $\alpha'\beta + \alpha\beta' - \alpha'\beta'$ (where these operations are performed in the surreal numbers, and are therefore natural operations on the corresponding ordinals). This expression is increasing in α' and β', since it can be written as $\alpha\beta - (\alpha - \alpha')(\beta - \beta')$. Therefore so is $f_{\alpha,\beta}(\alpha', \beta')$, the smallest ordinal greater than it. $\square$

Now, the proof:

Proof of Theorem 1.10. We split this into several cases depending on the value of α. If $\alpha \in \{0,1\}$ the theorem is obvious.

Now we have the case where $\alpha > 1$ is finite; in this case we will use Lemma 3.5 to give a computational proof. Let us rename α to a to make it clear that it is finite. Let $\beta = \beta' + b$ and $\gamma = \gamma' + c$ where β' and γ' are limit ordinals or 0, and b and c are finite.

So observe first that

$$\omega^{-1}(\beta' \oplus \gamma') = \omega^{-1}\beta' \oplus \omega^{-1}\gamma'.$$

This can be seen as, if $\beta' = \omega\beta''$ and $\gamma' = \omega\gamma''$, then

$$\omega(\beta'' \oplus \gamma'') = \omega\beta'' \oplus \omega\gamma'',$$

which can be seen by comparing Cantor normal forms. (This can also be seen by noting that for any ordinal δ, $\omega\delta = \omega \times \delta$, since if ε is a limit ordinal then $\varepsilon \oplus \omega = \varepsilon + \omega$, and by induction this quantity will always be a limit ordinal.)

Now, $\beta \oplus \gamma$ can be written as $(\beta' \oplus \gamma') + (b + c)$; here, $\beta' \oplus \gamma'$ is either 0 or a limit ordinal, and $b + c$ is finite. Thus,

$$a^{\otimes(\beta\oplus\gamma)} = \omega^{\omega^{-1}(\beta'\oplus\gamma')}a^{b+c} = \omega^{(\omega^{-1}\beta')\oplus(\omega^{-1}\gamma')}a^b a^c = (\omega^{\omega^{-1}\beta'}a^b) \otimes (\omega^{\omega^{-1}\gamma'}a^c) = a^{\otimes\beta} \otimes a^{\otimes\gamma},$$

as required.

This leaves the case where α is infinite. In this case we give an inductive proof, inducting on β and γ. If $\beta = 0$ or $\gamma = 0$ the theorem is obvious. If γ is a successor ordinal, say $\gamma = S\gamma'$, then

$$\alpha^{\otimes(\beta\oplus\gamma)} = \alpha^{\otimes(\beta\oplus S\gamma')} = \alpha^{\otimes S(\beta\oplus\gamma')} = \alpha^{\otimes(\beta\oplus\gamma')} \otimes \alpha = \alpha^{\otimes\beta} \otimes \alpha^{\otimes\gamma'} \otimes \alpha = \alpha^{\otimes\beta} \otimes \alpha^{\otimes\gamma},$$

as needed. If β is a successor, the proof is similar.

This leaves the case where β and γ are both limit ordinals. As before, not only are β and γ limit ordinals but so is $\beta \oplus \gamma$. So

$$\alpha^{\otimes\beta\oplus\gamma} = \sup\{\alpha^{\otimes\delta} : \delta < \beta \oplus \gamma\} = \sup(\{\alpha^{\otimes(\beta'\oplus\gamma)} : \beta' < \beta\} \cup \{\alpha^{\otimes(\beta\oplus\gamma')} : \gamma' < \gamma\}) \tag{3.8}$$

On the other hand,

$$\begin{aligned} \alpha^{\otimes\beta} \otimes \alpha^{\otimes\gamma} &= \sup\{f_{\alpha^{\otimes\beta},\alpha^{\otimes\gamma}}(\delta,\varepsilon) : \delta < \alpha^{\otimes\beta}, \varepsilon < \alpha^{\otimes\gamma}\} = \\ &\sup\{f_{\alpha^{\otimes\beta},\alpha^{\otimes\gamma}}(\alpha^{\otimes\beta'},\alpha^{\otimes\gamma'}) : \beta' < \beta, \gamma < \gamma'\} = \\ &\sup\{((\alpha^{\otimes\beta'} \otimes \alpha^{\otimes\gamma}) \oplus (\alpha^{\otimes\beta} \otimes \alpha^{\otimes\gamma'})) \ominus (\alpha^{\otimes\beta'} \otimes \alpha^{\otimes\gamma'}) : \beta' < \beta, \gamma' < \gamma\} = \\ &\sup\{(\alpha^{\otimes(\beta'\oplus\gamma)} \oplus \alpha^{\otimes(\beta\oplus\gamma')}) \ominus \alpha^{\otimes(\beta'\oplus\gamma')} : \beta' < \beta, \gamma' < \gamma\}. \end{aligned} \tag{3.9}$$

Note that here we have used not only the inductive hypothesis, but have also used Lemma 3.7 and the fact that $\alpha^{\otimes\gamma}$, $\alpha^{\otimes\beta}$, and their natural product are all limit ordinals.

So now once again we must show that the two sets we are taking the suprema of in the final expressions of Equations (3.8) and (3.9) are cofinal with each other. Let us call these sets S and T, respectively.

So let us take an element of S; say it is $\alpha^{\otimes(\beta'\oplus\gamma)}$ for $\beta' < \beta$. We want to show it is bounded above by some element of T. (If instead it is of the form $\alpha^{\otimes(\beta\oplus\gamma')}$ for $\gamma' < \gamma$, the proof is similar.) But certainly, choosing $\gamma' = 0$,

$$\alpha^{\otimes(\beta'\oplus\gamma)} \oplus \alpha^{\otimes\beta'} < \alpha^{\otimes(\beta'\oplus\gamma)} \oplus \alpha^{\otimes\beta}$$

and so

$$\alpha^{\otimes(\beta'\oplus\gamma)} < (\alpha^{\otimes(\beta'\oplus\gamma)} \oplus \alpha^{\otimes\beta}) \ominus \alpha^{\otimes\beta'}.$$

Conversely, say we take an element δ of T. Since we assumed α infinite, and in general we have $\deg(\alpha \otimes \beta) = (\deg\alpha) \oplus (\deg\beta)$, it follows that the sequence $\deg\alpha^{\otimes\beta}$ is strictly increasing in β. So here, we have an element δ of T given by $(\alpha^{\otimes(\beta\oplus\gamma')} \oplus \alpha^{\otimes(\beta'\oplus\gamma)}) \ominus \alpha^{\otimes(\beta'\oplus\gamma')}$ for some $\beta' < \beta$ and $\gamma' < \gamma$ and we want to determine its degree. Now, in general, if we have ordinals α and β, then $\deg(\alpha\oplus\beta) = \max\{\deg\alpha, \deg\beta\}$, and so it follows that if $\deg\alpha > \deg\beta$ then $\deg(\alpha\ominus\beta) = \deg\alpha$. So here it follows that

$$\deg\delta = \max\{\deg\alpha^{\otimes(\beta'\oplus\gamma)}, \deg\alpha^{\otimes(\beta\oplus\gamma')}\}.$$

But this means we can find an element of S with degree at least $\deg\delta$; and since β and γ are limit ordinals, we can find an element with degree even larger than $\deg\delta$, which in particular means that δ is less than some element of S.

Therefore S and T are cofinal and so have the supremum. This completes the proof. □

This then implies Theorem 1.11:

Proof of Theorem 1.11. We prove the more general version by induction on γ. If $\gamma = 0$, then

$$\alpha^{\otimes(\bigoplus_{i<0}\beta_i)} = \alpha^{\otimes 0} = 1 = \bigotimes_{i<0} \alpha^{\otimes\beta_i},$$

as needed.

If γ is a successor ordinal, say $\gamma = S\delta$, then

$$\alpha^{\otimes(\bigoplus_{i<S\delta}\beta_i)} = \alpha^{\otimes((\bigoplus_{i<\delta}\beta_i)\oplus\beta_\delta)} = \alpha^{\otimes(\bigoplus_{i<\delta}\beta_i)} \otimes \alpha^{\otimes\beta_\delta} = \bigotimes_{i<\delta}(\alpha^{\otimes\beta_i}) \otimes \alpha^{\otimes\beta_\delta} = \bigotimes_{i<S\delta}\alpha^{\otimes\beta_i},$$

again as needed, where we have applied both Theorem 1.10 and the inductive hypothesis.

Finally, if γ is a limit ordinal, so $\gamma = \lim_{\delta<\gamma}\delta$, then

$$\alpha^{\otimes(\bigoplus_{i<\gamma}\beta_i)} = \alpha^{\otimes(\lim_{\delta<\gamma}\bigoplus_{i<\delta}\beta_i)} = \lim_{\delta<\gamma}\alpha^{\otimes(\bigoplus_{i<\delta}\beta_i)} = \lim_{\delta<\gamma}\bigotimes_{i<\delta}\alpha^{\otimes\beta_i} = \bigotimes_{i<\gamma}\alpha^{\otimes\beta_i},$$

where here we have used both the inductive hypothesis and the fact that $\alpha^{\otimes\beta}$ is continuous in β (a fact which follows immediately from the definition).

The restricted version then follows by letting $\beta_i = \beta$ for all i. □

Thus we see that super-Jacobsthal exponentiation admits algebraic laws similar to those followed by ordinary exponentiation and Jacobsthal exponentiation.

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Appendix A. Comparisons between the operations

In Table 1 it was asserted that each operation appearing in the table is pointwise less-than-or-equal-to those appearing to the right of it in the table. In this appendix we justify that assertion. Let us state this formally:

Proposition A.1. *For any ordinals α and β, one has:*

(1) $\alpha + \beta \le \alpha \oplus \beta$
(2) $\alpha\beta \le \alpha \times \beta \le \alpha \otimes \beta$
(3) $\alpha^{\beta} \le \alpha^{\times\beta} \le \alpha^{\otimes\beta}$

The inequalities $\alpha + \beta \le \alpha \oplus \beta$ and $\alpha\beta \le \alpha \otimes \beta$ are well known; the inequalities $\alpha\beta \le \alpha \times \beta$ and $\alpha^{\beta} \le \alpha^{\times\beta}$ are due to Jacobsthal [11]. We will give proofs of all of the above nonetheless.

Proof. First we prove that $\alpha + \beta \le \alpha \oplus \beta$, by induction on β. If $\beta = 0$, both sums are equal to α. If $\beta = S\gamma$, then by the inductive hypothesis,

$$\alpha + \beta = S(\alpha + \gamma) \le S(\alpha \oplus \gamma) = \alpha \oplus \beta.$$

Finally, if β is a limit ordinal, then since $\alpha \oplus \beta$ is increasing in β, we have that

$$\alpha \oplus \beta \ge \sup_{\gamma<\beta}(\alpha \oplus \gamma) \ge \sup_{\gamma<\beta}(\alpha + \gamma) = \alpha + \beta.$$

So $\alpha + \beta \le \alpha \oplus \beta$. It then immediately follows from transfinite induction and the definitions of each that $\alpha\beta \le \alpha \times \beta$, and $\alpha^{\beta} \le \alpha^{\otimes\beta}$.

Next we prove that $\alpha \times \beta \le \alpha \otimes \beta$, again by induction on β. If $\beta = 0$, both products are equal to 0. If $\beta = S\gamma$, then by the inductive hypothesis,

$$\alpha \times \beta = (\alpha \times \gamma) \oplus \alpha \le (\alpha \otimes \gamma) \oplus \alpha = \alpha \otimes \beta.$$

Finally, if β is a limit ordinal, then since $\alpha \otimes \beta$ is (possibly weakly) increasing in β, we have that

$$\alpha \otimes \beta \ge \sup_{\gamma<\beta}(\alpha \otimes \gamma) \ge \sup_{\gamma<\beta}(\alpha \times \gamma) = \alpha \times \beta.$$

So $\alpha \times \beta \le \alpha \otimes \beta$. It then immediately follows from transfinite induction and the definitions of each that $\alpha^{\times\beta} \le \alpha^{\otimes\beta}$. This completes the proof. $\square$

All the above inequalities can of course also be proven by comparing Cantor normal forms. The inequalities $\alpha + \beta \le \alpha \oplus \beta$ and $\alpha\beta \le \alpha \otimes \beta$ also both follow immediately from the order-theoretic interpretation of each. Lipparini's order-theoretic interpretation [14] of $\alpha \times \beta$ also makes it clear that $\alpha\beta \le \alpha \times \beta$, although it does not seem to immediately prove that $\alpha \times \beta \le \alpha \otimes \beta$. And, of course, all these inequalities hold equally well for the infinitary versions of these operations.

Remark A.2. Note that if we were to consider the surreal exponentiation $(\alpha, \beta) \mapsto \exp(\beta \log \alpha)$, where exp is as defined by Gonshor [7, pp. 143–190], the same style of argument used above to prove $\alpha + \beta \le \alpha \oplus \beta$ and $\alpha \times \beta \le \alpha \otimes \beta$ could also be used

to prove $\alpha^{\otimes\beta} \leq \exp(\beta \log \alpha)$, in accordance with Table 1. But again, $\exp(\beta \log \alpha)$ is typically not an ordinal.

References

[1] U. Abraham and R. Bonnet, Hausdorff's theorem for posets that satisfy the finite antichain property, *Fund. Math.* **159** (1999), 51–69.

[2] D. Asperó and K. Tsaprounis, Long Reals, `https://archive.uea.ac.uk/ bfe12ncu/long_reals3.pdf`, 2015

[3] H. Bachmann, *Transfinite Zahlen*, Second Edition, Springer-Verlag, Berlin/Heidelberg, 1967.

[4] P. W. Carruth, Arithmetic of ordinals with applications to the theory of ordered abelian groups, *Bull. Amer. Math. Soc.* **48** (1942), 262–271.

[5] J. H. Conway, *On Numbers and Games*, Second Edition, A K Peters, Ltd., Natick, Massachusetts, 2001.

[6] D. H. J. De Jongh and R. Parikh, Well-partial orderings and hierarchies, *Indag. Math.* **39** (1977), 195–206.

[7] H. Gonshor, *An Introduction to the Theory of Surreal Numbers*, Cambridge University Press, Cambridge, 1986.

[8] H. Gonshor, Number Theory for the Ordinals With a New Definition for Multiplication, *Notre Dame Journal of Formal Logic* **21** (1980), 708–710.

[9] F. Hausdorff, *Set Theory*, trans. J. R. Aumann et. al., Second Edition, Chelsea Publishing Company, New York, 1962.

[10] J. Hickman, A Note on Conway Multiplication of Ordinals, *Notre Dame Journal of Formal Logic* **24** (1983), 143–145.

[11] E. Jacobsthal, Zur Arithmetik der transfiniten Zahlen, *Math. Ann.* **67** (1909), 130–143.

[12] P. Keddie, Ordinal Operations on Surreal Numbers, *Bull. London Math. Soc.* **26** (1994), 531–538.

[13] P. Lipparini, An infinite natural sum, `arXiv:1501.05123`, 2015

[14] P. Lipparini, Some iterated natural sums, `arXiv:1509.04078`, 2015